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A general harmonic coordinate transformation to simulate the states of strain in inhomogeneously deformed rocks

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Abstract—In order to investigate the possible states of strain that may exist in an inhomogeneously deformed body, a general deformation is proposed in the form of a harmonic coordinate transformation. This deformation differs from those previously considered by Jaeger, Ramsay & Graham and Hobbs in that it is capable of representing a wide variety of deformations; previous efforts had inbuilt assumptions regarding the mechanism of deformation. The transformation contains adjustable terms, all of which have distinct geometrical significance; some represent a homogeneous deformation, some represent inhomogeneous shortening, some represent inhomogeneous shear and others correspond to a 'pinch and swell' type of deformation. By combining these terms with different degrees of emphasis many kinds of deformation may be simulated. In this paper two constraints are developed in conjunction with the general harmonic transformation; these are the conditions for constant volume deformation (both locally and generally) and the condition for zero shear strain of lines initially normal to the distorted layer (again both locally and generally).

INTRODUCTION

A TRULY benevolent creator would have implanted a regular three-dimensional grid in rocks prior to their deformation that would enable us now to specify the strain at each point in a deformed rock (Fig. 1). However, He chose to set a problem by providing only parts of that grid; for example, regular foliation surfaces that commonly may be assumed to have been planar and parallel prior to deformation and a lineation which commonly may be assumed to have been linear (Fig. 2). Clearly there is not enough information in this incomplete grid to specify the strain at each point in a deformed body but the information that is provided does place constraints on the states of strain that exist. The aims of this paper are to explore these constraints and to discuss the other kinds of information that are necessary in order to arrive at a complete specification of strain in deformed rocks.

The problem set by the hypothetical benevolent creator could be solved in the following manner: In the undeformed state (Fig. 1a) a system of coordinates, $\mathbf{X}$, is erected, defined by the normal to the lineation in the foliation (X^1) the lineation (X^2) and the normal to the foliation (X^3). In the deformed state (Fig. 1b), another

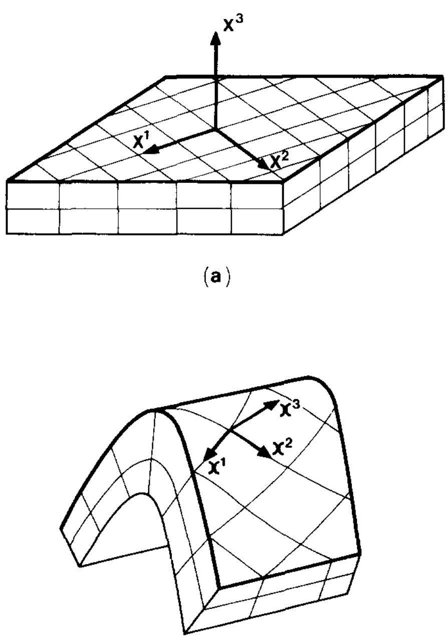

Fig. 1. Inhomogeneous deformation of an initial system of coordinate axes $\mathbf{X}$ to become x in the deformed state. Given the coordinate transformation from $\mathbf{X}$ to x, the state of strain is defined at each point.

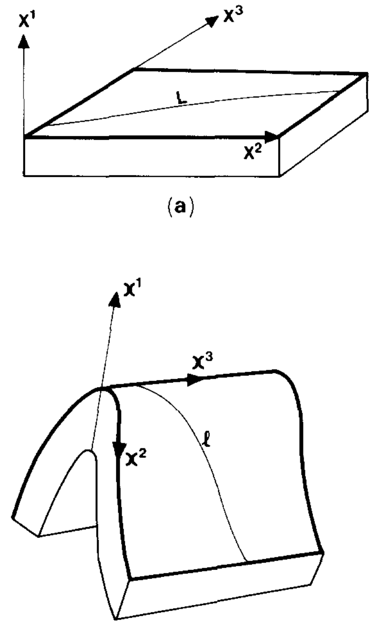

Fig. 2. Situation common in rocks where the material lines parallel to X^1, X^2, X^3 in the undeformed state are not necessarily recognizable in the deformed state. An arbitrary system of coordinates x^1, x^2, x^3 is defined for the deformed state. Again if the coordinate transformation from $\mathbf{X}$ to x is known then the state of strain is defined at each point.

system of coordinates (x) is erected corresponding to the deformed equivalents of the material lines in the undeformed state. In general this new coordinate system is non-orthogonal and curvilinear.

It is then possible to define the deformation by a coordinate transformation

$$\begin{aligned} x^1 &= x^1 (X^1, X^2, X^3) \\ x^2 &= x^2 (X^1, X^2, X^3) \\ x^3 &= x^3 (X^1, X^2, X^3) \end{aligned} \tag{1}$$

which simply says that the coordinates of a point in the deformed state are functions (linear or non-linear) of the undeformed coordinates.

The strain at each point in the deformed state is then given by

$$c^{-1^{kl}} = G^{\mathrm{KL}} \frac{\partial x^k}{\partial X^{\mathrm{K}}} \frac{\partial x^l}{\partial X^{\mathrm{L}}} \tag{2}$$

where G^{KL} is the metric tensor for the X^{K} coordinate system; since this system is Cartesian, $G^{\mathrm{KL}} = \delta^{\mathrm{KL}}$, the Kronecker delta. The Finger tensor, $\mathbf{c}^{-1}$, is chosen here because it gives the strain relative to the deformed state; its proper numbers are the squares of the principal stretches (see Truesdell & Toupin 1960, Eringen 1962).

As indicated, in real rocks an incomplete marker system is provided but a coordinate system can be erected in the deformed state, one possibility being as indicated in Fig. 2. The deformation is then defined by a coordinate transformation which has the general form of equation (1) and the strain at each point is defined by (2). However, the problem now is to discover the nature of the coordinate transformation (1) that describes the deformation. This problem has been tackled in the past by Jaeger (1969), Ramsay & Graham (1970) and Hobbs (1971). These authors proposed various kinds of coordinate transformations and proceeded to calculate strain distributions. However, the transformations used had inbuilt assumptions regarding the mode of deformation. Jaeger (1969) discussed both similar and concentric folds but the transformations used had the inherent assumptions that similar folds develop by shearing parallel to the axial surface and that concentric folds involve bending. The Ramsay & Graham (1970) transformations lead to similar folds and again presuppose shearing parallel to the axial surface as the basic mode of development. Hobbs (1971) considered Class 1, 2 and 3 folds of Ramsay (1967) but again prescribed bending (or bending plus flattening) as important parts of the development of Class 1 folds, and shearing with homogeneous and inhomogeneous shortening, respectively, as important mechanisms involved in the development of Class 2 and 3 folds.

Although such assumptions regarding folding mechanisms may be justified in some instances, it is clearly desirable to define the deformation as far as possible by means of a coordinate transformation that does not have a specific folding mechanism inherently built in.

In this paper, such a general deformation is presented in the form of a harmonic transformation which contains the various elements of a general inhomogeneous deformation (bending, shearing, shortening, pinch and swell). These components may be assembled in different proportions by a suitable choice of coefficients.

We examine first the nature of this general harmonic transformation and use it to consider states of strain in two dimensions in the profile planes of folds. We then proceed to consider other information that might further constrain the states of strain in deformed rocks. This paper forms part of a much wider analysis of coordinate transformations made by Hirsinger (1976a), part of which has been applied to deformed fossils (Hirsinger 1976b). The problem of gaining information from redistributed lineations is not considered here but will form the subject of a subsequent paper.

A GENERAL HARMONIC TRANSFORMATION APPLIED TO FOLDS

In this section we consider a general two-dimensional transformation aimed at specifying the strain within profile planes of folds. We consider only Cartesian reference frames in the strained and unstrained states (Fig. 3) although, as has already been indicated, the approach may readily be extended to non-Cartesian frames (see also Hobbs 1971).

The coordinate axes, X^{K}, in the undeformed state are taken normal and parallel to the surface to be folded (Fig. 3) and the coordinate axes, x^k, in the deformed state are taken parallel and normal to the axial plane.

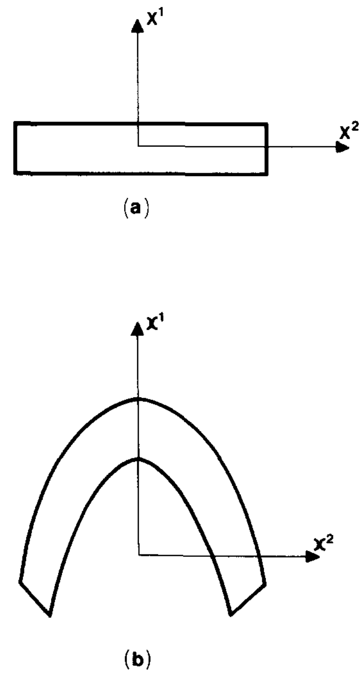

Fig. 3. Coordinate systems adopted in this paper. X^1 in the undeformed state is normal to the layering and X^2 is parallel. In the deformed state x^1 is parallel to the axial plane and normal to the fold axis; x^2 is normal to the axial plane. Both $\mathbf{X}$ and x are taken as Cartesian systems.

The transformation expressing the deformation is

$$\begin{aligned} x^i = a_{iK}X^K &+ \sum_m \sum_n \{A_{imn}\cos(mw_1X^1)\cos(nw_2X^2) \\ &+ B_{imn}\cos(mw_1X^1)\sin(nw_2X^2) \\ &+ C_{imn}\sin(mw_1X^1)\cos(nw_2X^2) \\ &+ D_{imn}\sin(mw_1X^1)\sin(nw_2X^2)\} \end{aligned} \tag{3}$$

where $w_K = 2\pi/\lambda_K$, λ_K being the fundamental wavelength of the inhomogeneous deformation in the X^K direction and a_{iK}, A_{imn}, B_{imn}, C_{imn}, D_{imn} are various coefficients, the significance of which is considered below.

Redundant coefficients

The transformation (3) is summed over all terms from $m = 0$, $n = 0$ onwards. As a result, sin(0) appears in some terms making the corresponding coefficients redundant. These are B_{im0}, C_{i0n}, D_{im0} and D_{i0n}.

Number of terms

If m is summed to M and n is summed to N then the number of non-redundant terms, T, is given by

$$\begin{aligned} T = 4 \quad & (\text{the } a_{iK} \text{ terms}) \\ + 2 \quad & (A_{100} \text{ and } A_{200} \text{ terms}) \\ + 4M \quad & (A_{1m0}, C_{1m0}, A_{2m0}, C_{2m0} \text{ terms}) \\ + 4N \quad & (A_{10n}, B_{10n}, A_{20n}, B_{20n} \text{ terms}) \\ + 8MN \quad & (A_{imn}, B_{imn}, C_{imn}, D_{imn} \text{ terms for} \\ & m \neq 0, n \neq 0, i = 1, 2). \end{aligned} \tag{4}$$

Thus, for $M = 6, N = 4$, there are 238 non-redundant terms. In such a case there are six non-zero harmonics along the X^1 axis and four non-zero harmonics along the X^2 axis; the series is said to be of order 6 × 4.

Homogeneous deformation—the a_{iK} *terms*

Setting all coefficients equal to zero except for the a_{iK} terms gives the transformation

$$\begin{aligned} x^1 &= a_{11}X^1 + a_{12}X^2 \\ x^2 &= a_{21}X^1 + a_{22}X^2. \end{aligned} \tag{5}$$

This represents a homogeneous deformation in which a_{11}, a_{22} are shortening or extension components and a_{12}, a_{21} are shear components. Figure 4(a) shows a deformation for which $a_{11} = a_{22} = 1$ and $a_{12} = -0.3$, $a_{21} = +0.6$. Figure 4(b) shows a pure shear deformation with $a_{11} = 0.5$, $a_{22} = 2.0$ and $a_{12} = a_{21} = 0$.

Rigid-body translation, the A_{i00} *terms*

Substituting $m = 0$, $n = 0$ into the general transformation (3) results in zero for all terms except the a_{iK} and A_{100}, A_{200}. Thus, a constant is added to the homogeneous transformation (5). Hence, the A_{i00} terms represent a rigid transport of the deforming body. Since the derivatives of these terms are zero they do not enter into expression (2) for the Finger tensor and hence need not be considered in any analysis of strain.

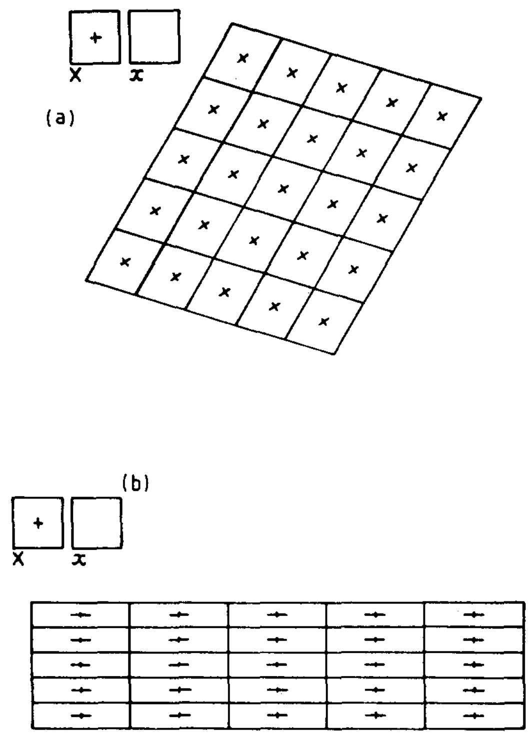

Fig. 4. Homogeneous transformations, in which all the harmonic terms are zero. Values of a_{iK} shown: (a) $a_{11} = 1$, $a_{22} = 1$, $a_{12} = -0.3$, $a_{21} = 0.6$; (b) $a_{11} = 0.5$, $a_{22} = 2$, $a_{12} = 0$, $a_{21} = 0$. Undeformed grid elements of the coordinate systems **X** and $\boldsymbol{x}$ superimposed in the undeformed and deformed states, respectively are plotted at the top left corner of each diagram.

Inhomogeneous shear in one direction, A_{i0n}, B_{i0n}, A_{2m0}, C_{2m0} *terms*

If A_{10n}, B_{10n} are taken as non-zero with A_{2m0}, C_{2m0} equal to zero then the general transformation becomes

$$\begin{aligned} x^1 &= a_{11}X^1 + a_{12}X^2 \\ &\quad + \sum_{n=1}^{N} \{A_{10n}\cos(w_2nX^2) + B_{10n}\sin(w_2nX^2)\} \\ x^2 &= a_{21}X^1 + a_{22}X^2. \end{aligned} \tag{6}$$

Such a transformation represents an inhomogeneous simple shear along the x^1-axis. Similar results hold for $A_{10n} = B_{10n} = 0$ and A_{2m0} and C_{2m0} non-zero, which represents an inhomogeneous simple shear along the x^2-axis. In both cases a homogeneous deformation is superimposed on the simple shear. Examples are shown in Fig. 5. When a_{iK} assumes the value of the Kronecker delta, δ_{iK}, expression (6) reduces to the type of series used for example for layer shape analysis by Hudleston (1973).

Since the inhomogeneity expressed by (6) is a function of one variable only, the resulting pattern of deformation is repeated identically parallel to one set of lines. Folds will therefore be similar in style. Three examples with varying degrees of overall shear are shown in Fig. 6. For Fig. 6(a), $a_{iK} = \delta_{iK}$ and the other coefficients are $A_{101} = -0.5$, $A_{102} = 0.1$, $B_{101} = -1.3$, $B_{102} = 0.3$, $B_{103} = -0.2$. Figure 6(b) represents the same transformation but with a pure shear component superimposed,

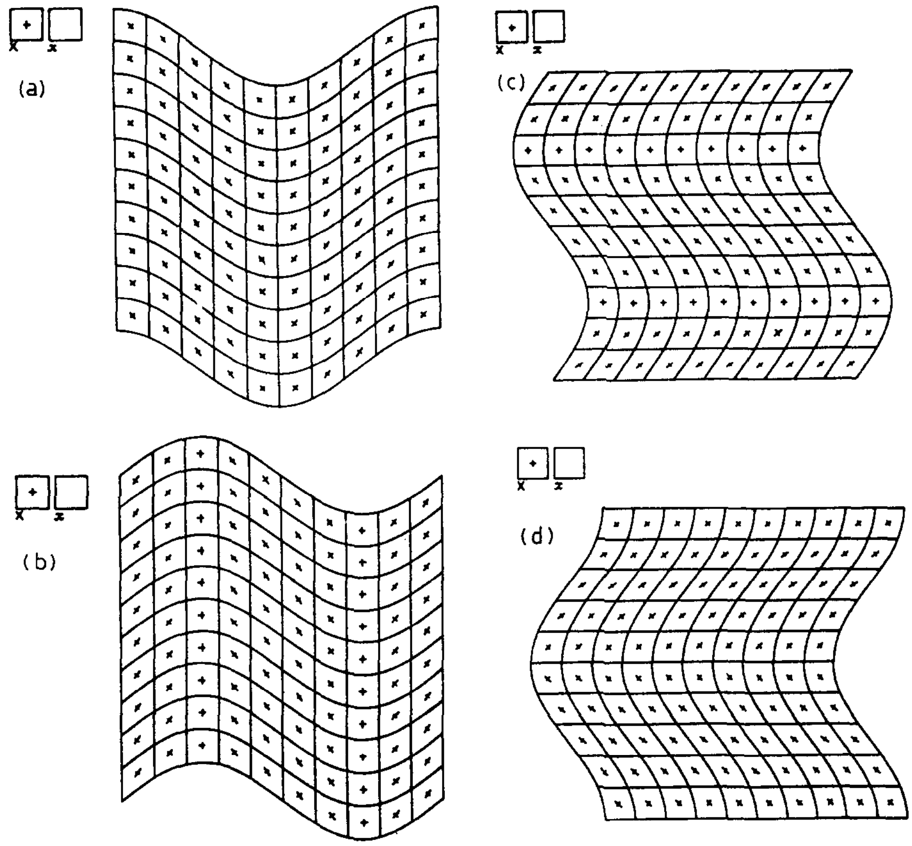

Fig. 5. Single harmonic transformations plotted in the range $0 \leqslant X^1 \leqslant 2\pi$ and $0 \leqslant X^2 \leqslant 2\pi$. Harmonic values used are: (a) $A_{101} = 0.75$, (b) $B_{101} = 0.75$, (c) $C_{210} = 0.75$ and (d) $D_{210} = 0.75$ with all other harmonics at zero, and $a_{iK} = \delta_{iK}$.

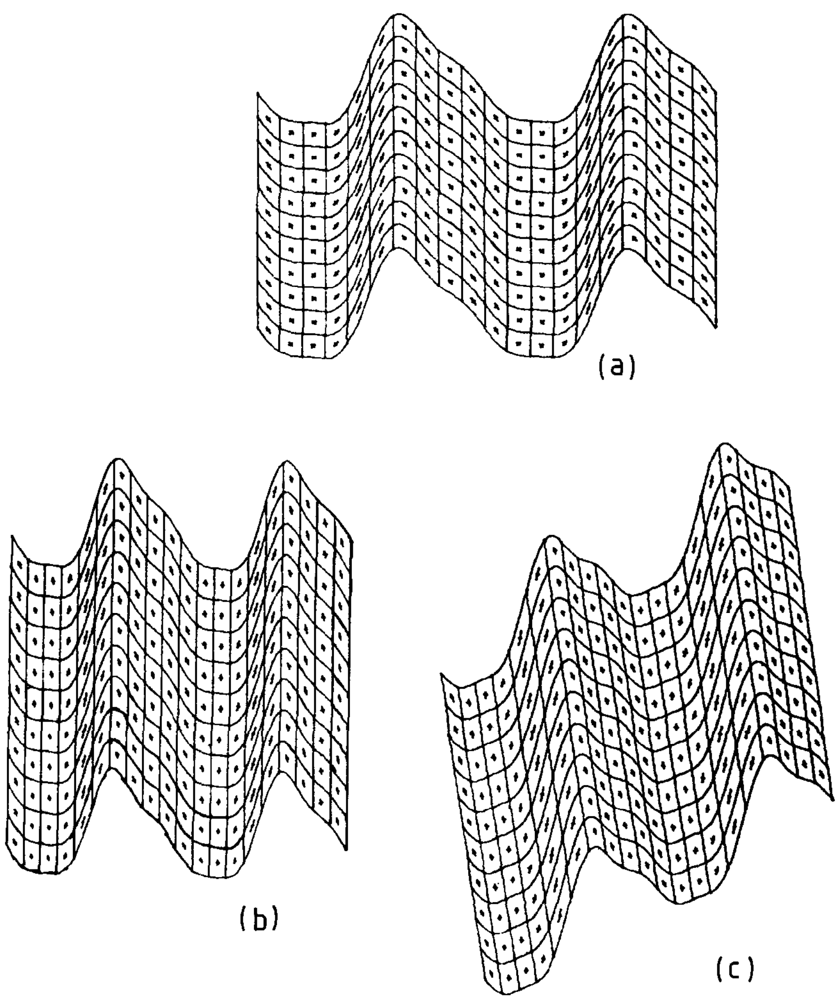

Fig. 6. An inhomogeneous simple shear transformation, (a) plotted without overall homogeneous shear, (b) with a pure shear of $a_{11} = 1.33$, $a_{22} = 0.75$, and (c) with this plus an overall simple shear given by $a_{12} = 0.4$ and $a_{21} = -0.2$. The areas plotted lie in the range $(0 \leqslant X^1 \leqslant 2\pi,\ 0 \leqslant X^2 \leqslant 4\pi)$. Inhomogeneous components are given by the harmonics: $A_{101} = -0.5$, $A_{102} = 0.1$, $B_{101} = -1.3$, $B_{102} = 0.3$ and $B_{103} = -0.2$.

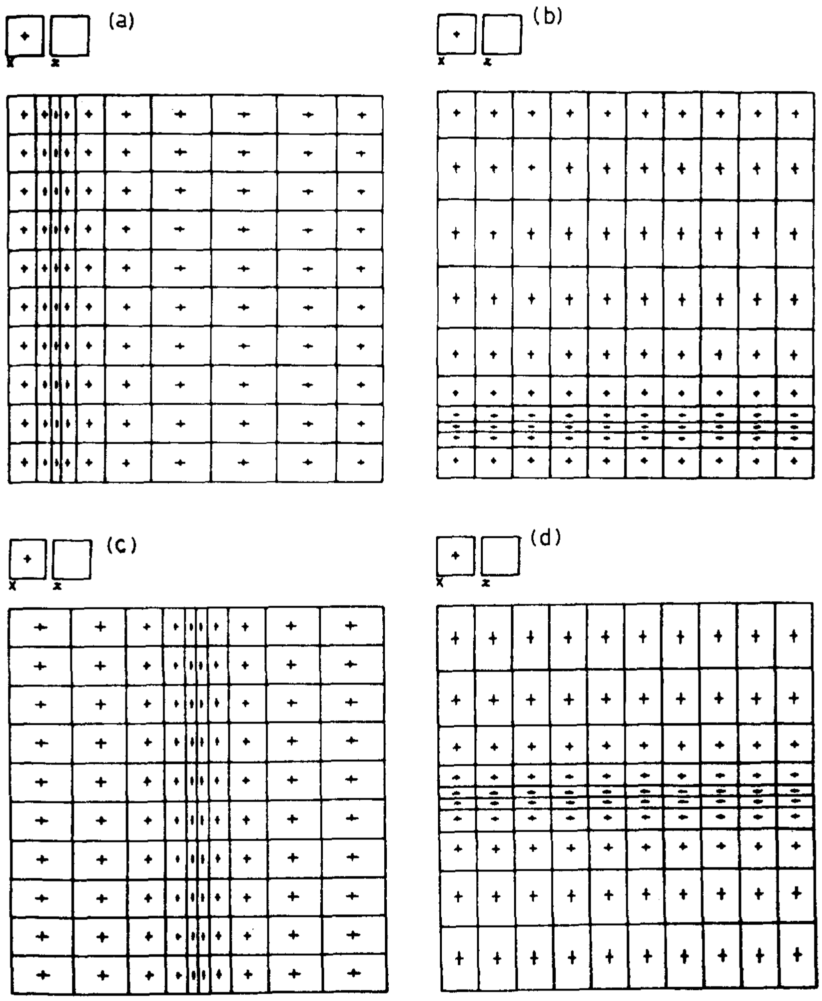

Fig. 7. Individual transformations showing the effects of the four possible first-order axially inhomogeneous pure shear harmonics, plotted in the range ($0 \leq X^1 \leq 2\pi$, $0 \leq X^2 \leq 2\pi$), with $a_{iK} = \delta_{iK}$. All harmonics are zero except for: (a) $A_{201} = 0.75$, (b) $A_{110} = 0.75$, (c) $B_{201} = 0.75$ and (d) $C_{110} = 0.75$.

defined by $a_{11} = 1.33$, $a_{22} = 0.75$, $a_{12} = a_{21} = 0$. Figure 6(c) illustrates the effect of adding shear components, $a_{12} = 0.4$, $a_{21} = -0.2$, in addition to the pure shear of Fig. 6(b).

Inhomogeneous pure shear parallel to coordinate axes, A_{1m0}, C_{1m0}, A_{20n}, B_{20n} *terms*

The remaining terms involving coefficients with some zero in their specification, result in fluctuations in the position of one set of coordinates relative to the corresponding initial coordinates and thus lead to an inhomogeneous pure shear. Considering, for instance, the X^2 axis only; then an appropriate transformation would be one having A_{20n} and B_{20n} non-zero and $A_{1m0} = C_{1m0} = 0$ giving

$$\begin{aligned} x^1 &= a_{11}X^1 + a_{12}X^2 \\ x^2 &= a_{21}X^1 + a_{22}X^2 \\ &+ \sum_{n=1}^{N} \{A_{20n}\cos(w_2nX^2) + B_{20n}\sin(w_2nX^2)\}. \end{aligned} \tag{7}$$

Layers defined as planes of X^1 = constant would show no folding, the effect being a change in relative spacing between these planes. Figure 7 shows some examples. Notice the inhomogeneous volume change that accompanies deformations of this type.

Pinch and swell deformations, the mixed terms, A_{imm}, B_{imn}, C_{imn}, D_{imn} *with* $m \neq 0, n \neq 0$

Harmonics for which both m and n are non-zero give terms which are products of sines and cosines of two coordinates, and thus cause a more general form of strain inhomogeneity. This is a 'pinch and swell' type of deformation (Figs. 8 and 9), caused by inhomogeneous simple shear along one axis by an amount varying with respect to both axes. Alternatively, one could consider these harmonics as giving a 'differential flattening' as used by Ramsay (1962), albeit without the attendant fold formation which could maintain an isochoric pattern.

The addition of harmonics giving such variations both along X^1 and X^2 allows reduction of the dilations inherent in these terms, giving areas of deformation reminiscent of typical internal boudinage patterns (see Cobbold *et al.* 1971). Figure 10(a), for example, shows the transformation produced by having $A_{111} = 0.5$ and $A_{211} = 0.5$.

In considering the relationships of these harmonics to real deformations it must be remembered that the harmonics only give the inhomogeneous component of deformations and will not be realistic unless an overall homogeneous deformation is applied. Thus the pinch

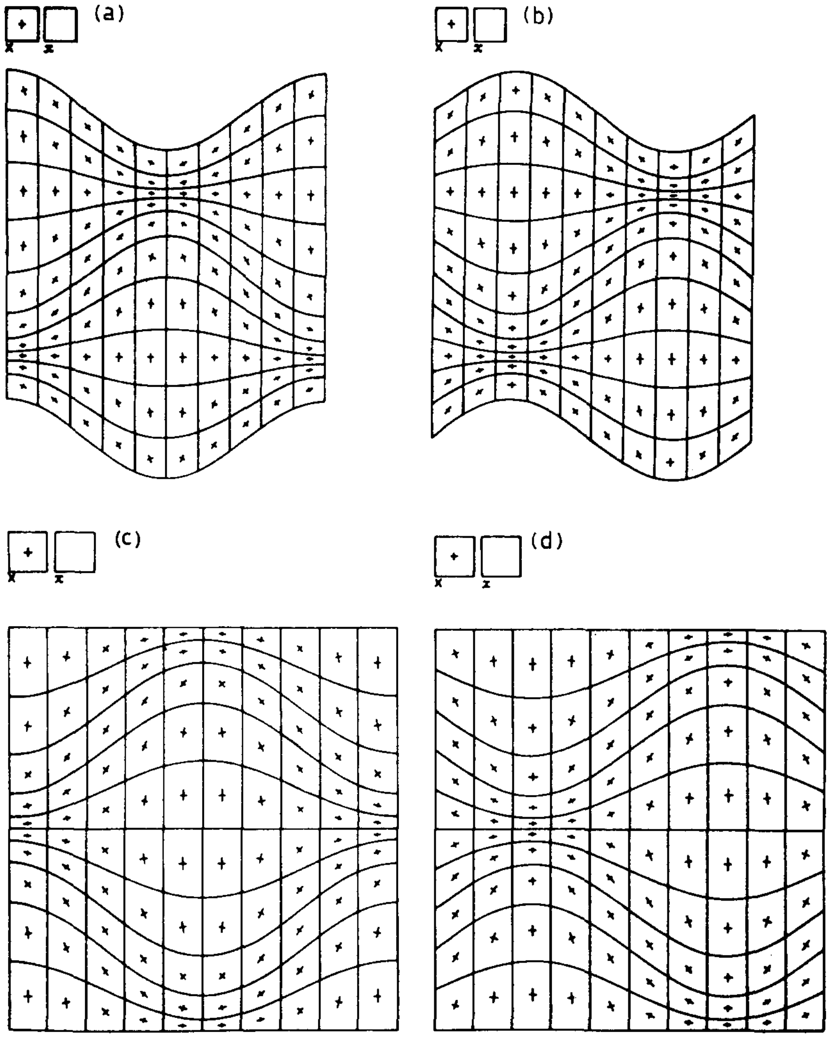

Fig. 8. 'Pinch-and-swell' transformations with single first order mixed harmonics on x^1 only, plotted over a full wavelength in both X^1 and X^2 ($0 \leq X^1 \leq 2\pi$, $0 \leq X^2 \leq 2\pi$). The harmonic values for individual diagrams are: (a) $A_{111} = 0.75$, (b) $B_{111} = 0.75$, (c) $C_{111} = 0.75$ and (d) $D_{111} = 0.75$ with all other harmonics at zero, and $a_{iK} = \delta_{iK}$.

and swell of Fig. 10(a) does not resemble boudinage until values of a_{iK} other than the Kronecker delta are applied. In Fig. 10(b) the values of $a_{11} = 0.7071$, $a_{22} = 1.414$ have been applied in addition to the harmonics used for Fig. 10(a), such that virtually all the strain in the x^2 direction (vertical) becomes a shortening.

It is by the application of this form of harmonic on top of the harmonics used to achieve fold forms due to inhomogeneous simple and pure shears that deformation patterns similar to those described as being due to 'buckling' can be produced. Thus if the transformation (6) for folding by inhomogeneous simple shear is combined with a component of deformation given by the harmonic A_{211} then alternating portions of the 'layering' become extended and shortened. In Fig. 11 such a result has been plotted for a limited range of X^1 only, the result being a folded layer with a strain pattern similar to that commonly related to buckling in geological structures (see for example the wax models of Cobbold 1975, fig. 11).

INTRODUCTION OF OTHER ASSUMPTIONS TO FURTHER CONSTRAIN STATES OF STRAIN

Although the transformation given by equation (3) is quite general and in principle is capable of generating all states of strain in simply folded rocks, not enough information is presented by just the geometry of the folded layer to specify the various coefficients in (3) uniquely. In real folds other information may be available, such as the state of strain at one or more places around the fold as revealed by deformed fossils, or it may be possible to assume that the orientation of one principal plane of strain is parallel to the trace of cleavage on the profile plane. All information of this type may be used to further constrain the range of strain states that are possible. Other assumptions may be of a more general nature; thus the deformation might be assumed to be isochoric (constant volume) or at least the dilational pattern might be assumed to be simple and to be restricted to a specific range of values. Another form of assumption may involve the type of strain that specific lines undergo. Thus, it may be reasonable to assume that the strain distribution, in layers folded by buckling, is such that lines originally straight and perpendicular to the layered boundary, remain normal to the layer after folding. This condition is referred to below as *shear strain neutrality*.

Clearly other assumptions are possible, but in this section only the two conditions of *isochorism* and of

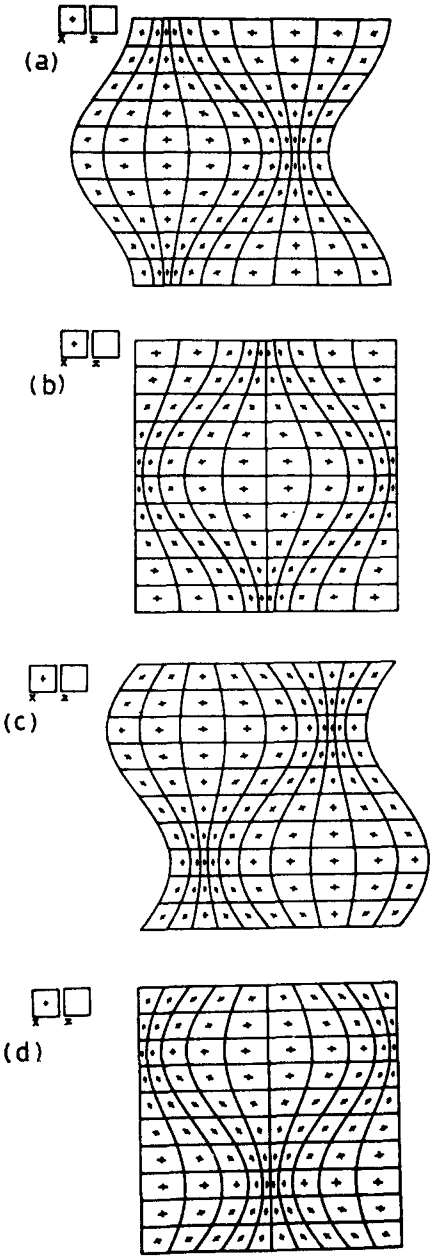

Fig. 9. As for Fig. 8, but illustrating mixed harmonics on x^2 instead of on x^1. Non-zero harmonics: (a) $A_{211} = 0.75$, (b) $B_{211} = 0.75$, (c) $C_{211} = 0.75$ and (d) $D_{211} = 0.75$.

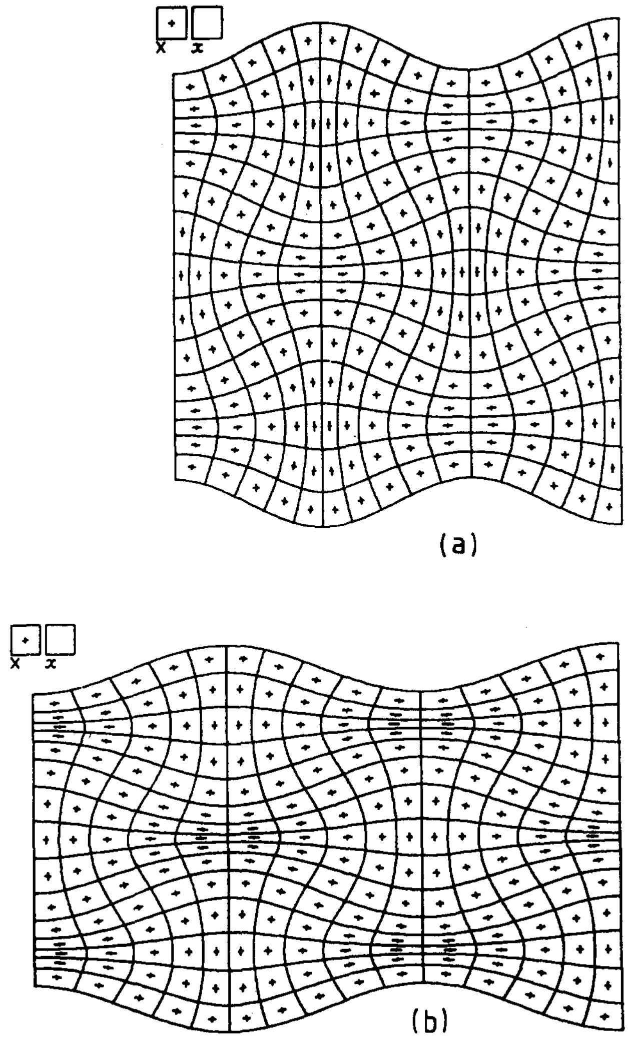

Fig. 10. Transformations produced by superposition of the harmonics $A_{111} = 0.5$ and $A_{211} = 0.5$, in the range $(-\pi \leq X^1 \leq 3\pi, -\pi \leq X^1 \leq 3\pi)$. (a) has $a_{iK} = \delta_{iK}$ (above), whilst (b) has an overall pure shear given by $a_{11} = 0.7071$, $a_{22} = 1.414$ (below).

shear strain neutrality are considered, in conjunction with the multiple harmonic transformation given by (3). The resulting expressions are readily quantifiable for specific choices of inhomogeneity. The analytical process by which transformations are generated to match the selected restrictions has been incorporated into a computer program, such that the derivation of the suitable equations becomes automatic, upon the user's selection of a starting point for the form of the transformation. Armed with this analytical aid, the effects of mixing various harmonic types, on the restrictions defined, can be explored.

In what follows, partial differential equations representing the restrictions of isochorism and of angular shear strain neutrality are derived, in terms of the derivatives of a general transformation. Specific solutions of these are then developed, and examples presented graphically.

Isochorism. For a general transformation of the form of equation (3), the ratio, J, of elemental volumes in the deformed and undeformed states is given by the determinant (see Hobbs 1971, p. 335)

$$J = \begin{vmatrix} \dfrac{\partial x^1}{\partial X^1} & \dfrac{\partial x^1}{\partial X^2} \\ \dfrac{\partial x^2}{\partial X^1} & \dfrac{\partial x^2}{\partial X^2} \end{vmatrix}. \tag{8}$$

For zero dilation, or for ischorism to exist between Cartesian coordinate systems X^K and x^k, (8) becomes

$$J = \begin{vmatrix} \dfrac{\partial x^1}{\partial X^1} & \dfrac{\partial x^1}{\partial X^2} \\ \dfrac{\partial x^2}{\partial X^1} & \dfrac{\partial x^2}{\partial X^2} \end{vmatrix} = 1. \tag{9}$$

Angular shear strain neutrality. This condition implies that there is no change in angle between two orthogonal

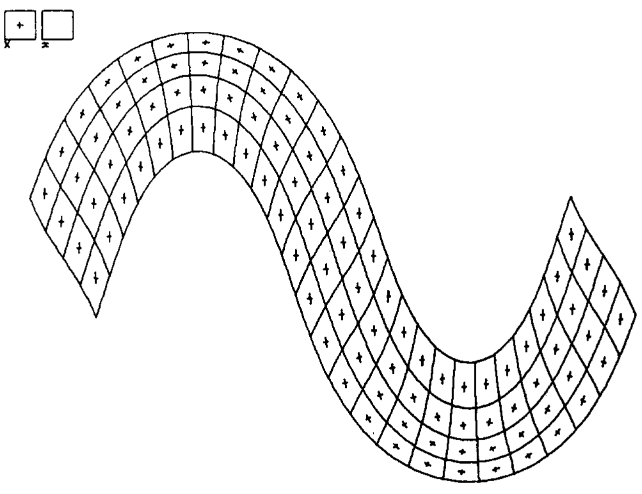

Fig. 11. A portion of a transformation in which elements of a 'buckling' type of deformation are represented. Parameters in the transformation are: $B_{101} = B_{111} = 1.0$, $C_{120} = -0.25$, $B_{202} = -0.15$, $C_{211} = -0.6$, $a_{11} = 1.414$, $a_{22} = 0.7071$, $w_1 = 1.0$ and $w_2 = 0.75$. The area of the transformation shown lies in the ranges $(-0.2\pi \leq X^1 \leq 0.2\pi)$ and $(0 \leq X^2 \leq 2.0\pi)$.

lines, undergoing the deformation being considered. In this section the condition will be applied to the X^1 and X^2 axial directions of a Cartesian coordinate system $\mathbf{X}$.

The change in angle, Γ, between originally orthogonal infinitesimal vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$, is given by (see Eringen 1962, p. 22)

$$\sin \Gamma(N_1,N_2) = C_{KL}\, N_1^K\, N_2^L / (\Lambda_{(N_1)}\, \Lambda_{(N_2)}) \qquad (10)$$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ are unit vectors parallel to $d\mathbf{X}_1$ and $d\mathbf{X}_2$, C_{KL} is Green's deformation tensor and $\Lambda_{(Ni)}$ is the stretch of $\mathbf{N}_i$. C_{KL} is given for Cartesian coordinate systems by

$$C_{KL} = \delta_{kl} \frac{\partial x^k}{\partial X^K} \frac{\partial x^l}{\partial X^L}. \qquad (11)$$

For no change in angle, $\sin \Gamma(N_1, N_2)$ must be zero. Hence this becomes the condition that

$$C_{KL} N_1^K N_2^L = 0. \qquad (12)$$

Now, since $\mathbf{N}_1$ and $\mathbf{N}_2$ are unit vectors parallel to Cartesian coordinate axes,

$$N_1^1 = 1,\ N_1^2 = N_2^1 = 0,\ N_2^2 = 1. \qquad (13)$$

Substitution of (11) into (12) gives

$$\delta_{kl} \frac{\delta x^k}{\delta X^K} \frac{\delta x^l}{\delta X^L} N_1^K N_2^L = 0. \qquad (14)$$

Substitution of the values of N from equation (13) gives, after appropriate cross-multiplication, the condition for angular shear strain neutrality along X^1 and X^2 as

$$\frac{\delta x^1}{\partial X^1} \frac{\partial x^1}{\partial X^2} + \frac{\partial x^2}{\partial X^1} \frac{\partial x^2}{\partial X^2} = 0. \qquad (15)$$

Method of solution of these restrictions

Incorporation of the derivatives of the general harmonic transformation (3) into an equation such as (9) or (15) defining strain restrictions, gives an expression which is the sum of terms that are products of the homogeneous strain parameters a_{iK}, the fundamental wavelength parameters w_K, the various harmonics A_{imn}, B_{imn}, etc., and geometrical identities in X^K representing the products of terms such as $\sin(mw_1X^1)\cos(nw_2X^2)$ and $\cos(mw_1X^1)\cos(nw_2X^2)$.

For a *general solution*, this expression must be valid for all values of X^K. In addition, a *limited solution*, such that the restrictions are satisfied along particular surfaces only, may be required. In such cases the solution is obtained whenever the expression is valid for the values of X^K conforming to such surfaces. The simplest case of this would be solutions for X^K = constant surfaces. For example, a transformation with no dilation in the hinge plane might be sought. For transformations in which folding results with axial planes given by X^2 = constant, hinge planes might be found at $X^2 = n\pi$, n being an integer. Appropriate solutions would thus be ones in which (9) holds for X^2 values that are integer multiples of π.

For an expression such as (15) to hold for all values of X^K, the simplest solution is found when the coefficients of the various geometric identities occurring as separate terms sum to zero, for each of these identities. When this occurs, parameter values have been found which are such that X^K values are arbitrary.

Similarly, if a limited solution is sought, then, in the case of X^K = constant, substitution of the appropriate X^K value into the geometric identities yields simpler geometric terms. For a solution, the sums of the individual groups of coefficients on these must sum to zero.

The most straight-forward approach is thus to take the general expression resulting from substitution of the appropriate derivatives into the restrictive equation, and rewrite it by grouping all the similar geometric identities individually. These geometric identity terms shall be called *variable terms* from here on. Each of the coefficient groups on particular terms now represents an equation in what shall now be called the *parameters*. These include the constant coefficients, the fundamental wavelength parameters, and the harmonic coefficients.

The fundamental difference between the parameters and variable terms is that the variable terms involve X^K, whereas the parameters are constant for a given transformation. The purpose of this exercise is to define groups of parameters, and hence transformations, which satisfy the restrictions imposed for the appropriate values of X^K.

Were the parameter groups simple, it might be possible to solve the coefficient groups as simultaneous equations. In fact, this is not generally possible. The problem is that we wish to start off with some harmonic terms specified, to give a general shape such as a fold, and then to find what additional harmonics must be included to produce an appropriately restricted complementary inhomogeneity. The most obvious case would be the imposition of a pinch and swell across the axial planes of similar folds, to produce a 'buckling' pattern. The parameters used do not fall into any particular combination of knowns and unknowns, and considerable examination and manipulation of parameters between individual equations is required for solution. In effect, the solution is one for a group of simultaneous equations in which individual equations have different numbers of terms, and involve products of parameters as well as linear terms.

The derivation and solution process is, despite this, quite mechanical, and must be performed for a large number of combinations of parameters to produce useful results. A computer program has been written to perform these tasks. Its results are used here without going through the intermediate steps of derivation. Details of the program are incorporated in Hirsinger (1976a).

Constant coefficients only

With all harmonic terms zero, the restriction for isochorism becomes

$$a_{11}a_{22} - a_{12}a_{21} = 1 \tag{16}$$

and that for shear strain neutrality becomes

$$a_{11}a_{12} + a_{21}a_{22} = 0. \tag{17}$$

First-order cosine curve

With a cosine curve, the presence of a pure shear component a_{11} alone is restrictive on the existence of shear strain neutrality; it is unaffected by the existence of non-zero simple shear components. A shear strain neutral surface along X^1 = constant is thus impossible unless additional inhomogeneity is present. This is predictable, as without it the transformation is compatible with the model of similar folding. There will always be a shear strain neutral surface along $X^2 = 0$. This simply represents the 'hinge' area of the cosine curve.

For isochorism, only the pure shear components figure in the homogeneous restrictions. Isochorism is destroyed, on the other hand, if any non-zero component of simple shear, a_{21}, is introduced. Selection of a_{12}, for any a_{11} and a_{22} and zero a_{21}, is arbitrary.

This is the crux of the problem of defining an isochoric buckling deformation using the harmonic series transformation (3). On the one hand, isochorism is readily attained using the similar fold transformation, whilst for the buckling effect, reflected in this case by attempting to get a neutral surface parallel to layering at some point, additional inhomogeneity must be present. Cases of inhomogeneity which might serve the purpose are now examined.

Cosine curve with a single mixed harmonic

The variation to produce shear strain neutrality will have to counteract the shear along x^1, and thus must represent a variation in x^2. Restrictions have been introduced resulting from the application of neutrality and

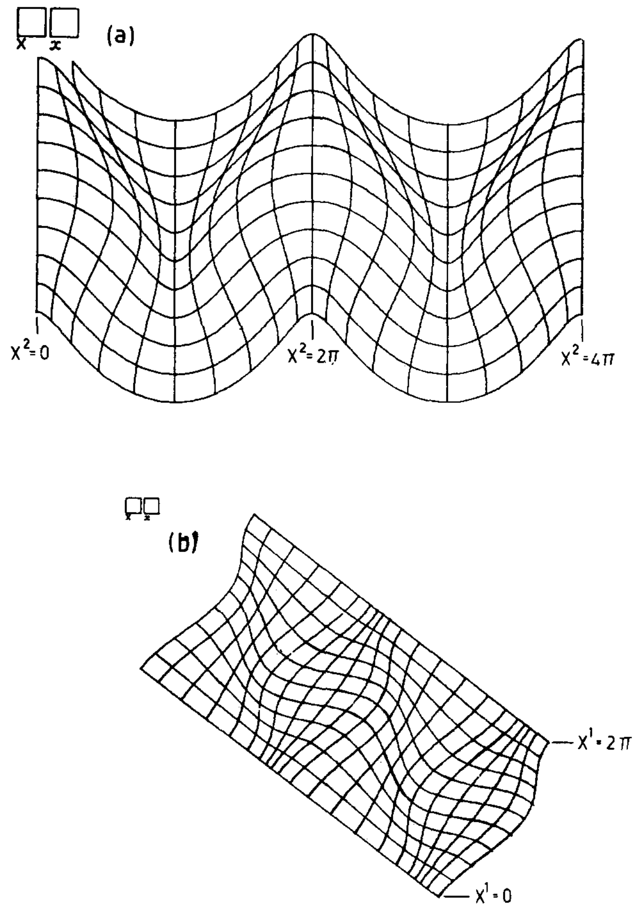

Fig. 12. Transformations with shear strain neutral surfaces along (a) $X^2 = 0$ and (b) $X^1 = 0$, plotted in the range ($0 \leqslant X^1 \leqslant 2\pi$, $0 \leqslant X^2 \leqslant 4\pi$), and defined by: (a) $A_{101} = 1$, $B_{211} = -0.5$, $a_{iK} = \delta_{iK}$; (b) $A_{101} = 0.5$, $A_{211} = -0.6666$, $a_{11} = a_{22} = 1$, $a_{12} = -0.75$ and $a_{21} = 0.75$.

isochorism conditions to combinations of the basic cosine curve A_{101} with pinch-and-swell type harmonics A_{211}, B_{211}, C_{211} and D_{211}, in turn.

Neither general isochorism (that is, isochorism at each point) nor general angular shear strain neutrality occurs for non-zero selection of these harmonics.

Shear strain neutral surfaces will exist along planes of $X^2 = 0$ for arbitrary combinations of A_{101} with B_{211} or D_{211}, but along the more desirable case of $X^1 = 0$ surfaces, only for specific cases, such that $a_{11}A_{101} = a_{21}A_{211}$.

Figure 12(a) illustrates neutral surfaces along $X^2 = 0$, in a transformation with unitary A_{101} and $B_{211} = -0.5$. Figure 12(b) shows a transformation with $A_{101} = 0.5$, $A_{211} = -0.6666$, a neutral surface along $X^1 = 0$ being achieved by solution of the special case mentioned above, the homogeneous strain parameters used being $a_1 = 1$, $a_{12} = -0.75$, $a_{21} = 0.75$ and $a_{22} = 1$. Note that the fold effect of A_{101} has been reduced by the effect of A_{211} with the simple shear terms, to an internal boudinage pattern.

Isochorism will also exist along specific layers, of X^2 = constant, for arbitrary choice of values of the harmonics, and also, for some specific cases (for example for $a_{11}A_{211} = a_{21}A_{101}$) along $X^1 = 0$. This is the opposite to the special case for neutrality along this layer, such

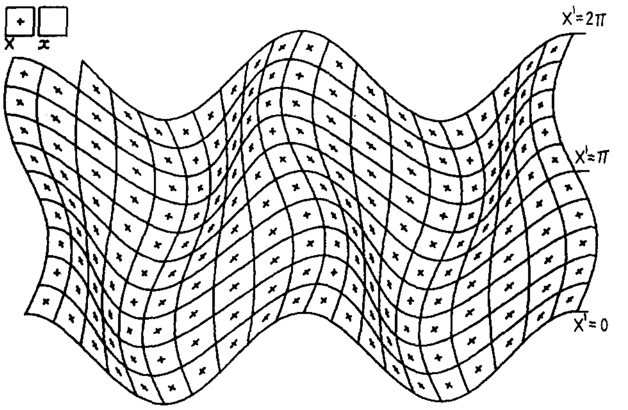

Fig. 13. A transformation that is isochoric along $X^2 = n\pi$, where n is any integer. The range of the plot is ($0 \leq X^1 \leq 2\pi$, $0 \leq X^2 \leq 4\pi$) such that these lines are at the far right, the far left, and in the 'hinge' regions. The transformation is defined by: $A_{101} = 1$, $C_{211} = 0.5$ and $a_{iK} = \delta_{iK}$.

that the two conditions are incompatible. Figure 13 shows a transformation having isochorism along $X^2 = 0$, namely one with $A_{101} = 1$, $C_{211} = 0.5$, and with homogeneous strain parameters equal to Kronecker delta.

Combinations of pinch and swell only

In the previous section combination of a simple cosine curve with a single mixed harmonic has been seen not to permit general isochorism. By combining additional pinch and swell harmonics such a solution might, nevertheless, eventually emerge. In the present section combinations of various pinch and swell harmonics, without an overall fold shape such as a cosine curve, are examined.

The conditions for neutrality with combinations of a single mixed harmonic on x^1 and another on x^2 have been examined for various combinations of A with A, A with D, D with D, B with B, B with C and C with C.

General shear strain neutrality is not possible for any of these cases, as the final equation in each represents a coefficient which is the sum of two squares, thus only going to zero for zero harmonics. It also appears that general isochorism is not possible either. However, there are many cases of shear strain neutral and isochoric surfaces. A few examples are presented here.

For combinations of A_{111} and D_{211} shear strain neutrality can be obtained along lines of $X^1 = 0$, provided that $w_2\, a_{11}\, A_{111} = w_1\, a_{22}\, D_{211}$. Figure 14(a) shows the case for Kronecker delta values to a_{iK}, unitary fundamental wavelength parameters, and $A_{111} = D_{211} = 0.5$. This transformation, incidentally, is also shear strain neutral along lines of $X^2 = 0.5\,\pi$.

In Fig. 14(b) a more complex case based on the same restrictions is shown, in which simple shear components have been included. The solution is for $a_{11} = 0.75$, $a_{12} = 0.5$, $a_{21} = -0.5$, $a_{22} = 0.75$, with A_{111} being specified as 0.75. $X^1 = 0$ is to be the shear strain neutral surface, the solution being given by $D_{211} = 0.75$. As in the previous case shear strain neutrality is maintained along $X^2 = 0.5\,\pi$. Note that since the components of simple shear, which correspond to those for a pure rotation, are added to the transformation, rather than being multiplied over the whole, Fig. 14(b) is not simply a sheared version of Fig. 14(a).

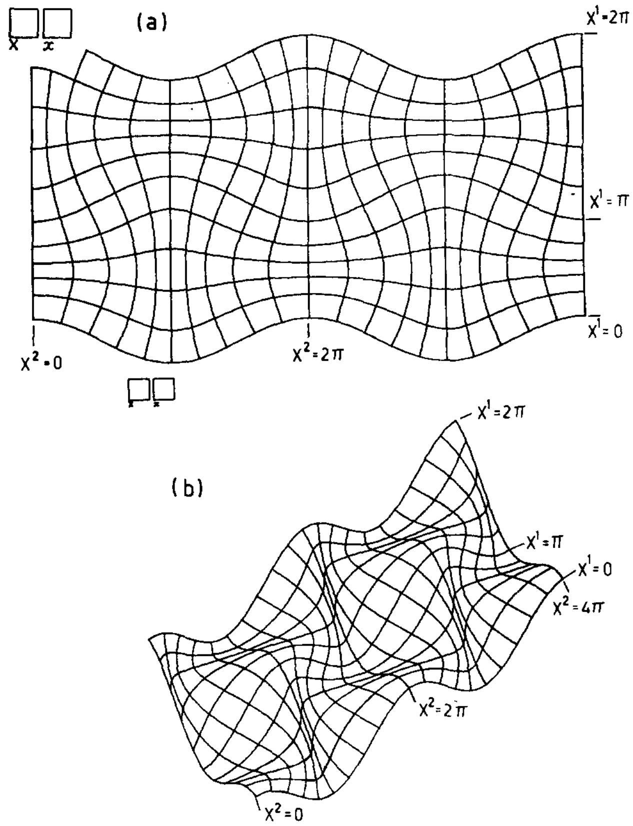

Fig. 14. Solutions for angular shear strain neutrality on $X^1 = n\pi$, plotted in the range ($0 \leq X^1 \leq 2\pi$, $0 \leq X^2 \leq 4\pi$). The transformations are: (a) $A_{111} = D_{211} = 0.5$; $a_{iK} = \delta_{iK}$ and (b) $A_{111} = D_{211} = 0.75$; $a_{11} = a_{22} = 0.75$; $a_{12} = 0.5$ and $a_{21} = -0.5$.

In Fig. 15 a similar exercise has been performed for isochorism. The transformation in Fig. 15(a) exhibits isochorism along $X^2 = 0.5\,\pi$, the solution being given by $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$, $A_{111} = 0.5$ and $D_{211} = -0.5$. The only difference from the shear strain neutral case is thus in the sign of D_{211}. The transformation is also isochoric for $X^1 = 0$, and neutrality occurs along $X^1 = 0.5\,\pi$.

The consideration of simple-shear terms in the isochoric case results in transformations such as, for example, that illustrated in Fig. 15(b), in which isochoric surfaces are along $X^1 = 0$ and $X^2 = 0.5\,\pi$. The transformation is given by $a_{11} = 0.75$, $a_{12} = 0.5$, $a_{21} = -0.5$, $a_{22} = 1$, $A_{111} = 0.75$ and $D_{211} = -0.75$. Note that the difference between this and its shear strain neutrality equivalent is greater than the case without simple shear.

An interesting case is that in which both the pinch and swell on X^1 and on X^2 are of the same 'harmonic type', that is, they are all A, all B, etc. For Fig. 16(a), shear strain neutrality was sought along $X^2 = 0$, starting off with $a_{12} = 0.5$, $a_{11} = 0.75$ and $a_{22} = 1$, with A_{111} being specified as 0.5 and A_{211} to be the complementary harmonic. It was determined that $a_{21} = -0.375$ and $A_{211} = -0.25$. The result of this combination is that

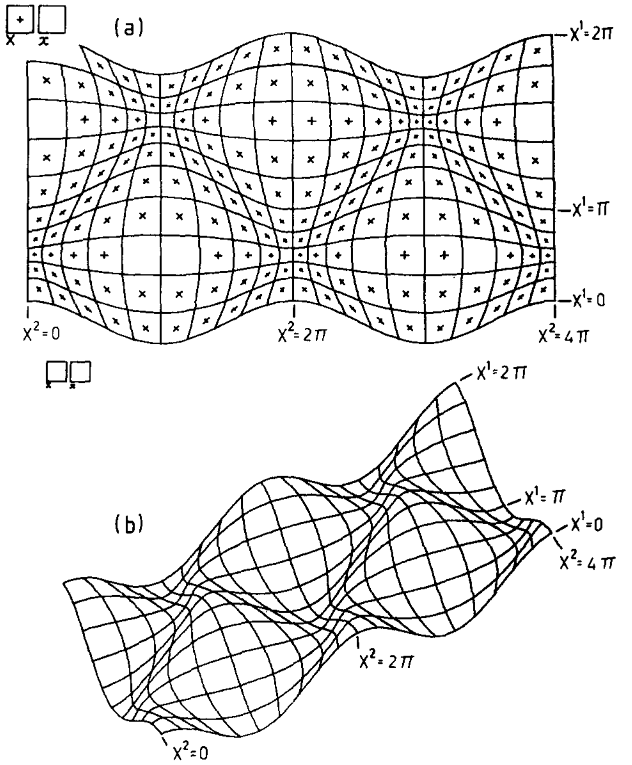

Fig. 15. Solutions for isochoric surfaces, (a) without and (b) with simple shear terms. Isochorism occurs along $X^1 = n\pi$ and $X^2 = (0.5 + n)\pi$ in both. The range of the plots is $(0 \leqslant X^1 \leqslant 2\pi, 0 \leqslant X^2 \leqslant 4\pi)$, the transformations being: (a) $A_{111} = 0.5$, $D_{211} = -0.5$ and $a_{iK} = \delta_{iK}$; (b) $A_{111} = 0.75$, $D_{211} = -0.75$, $a_{11} = 0.75$, $a_{12} = 0.5$, $a_{21} = -0.5$ and $a_{22} = 1$.

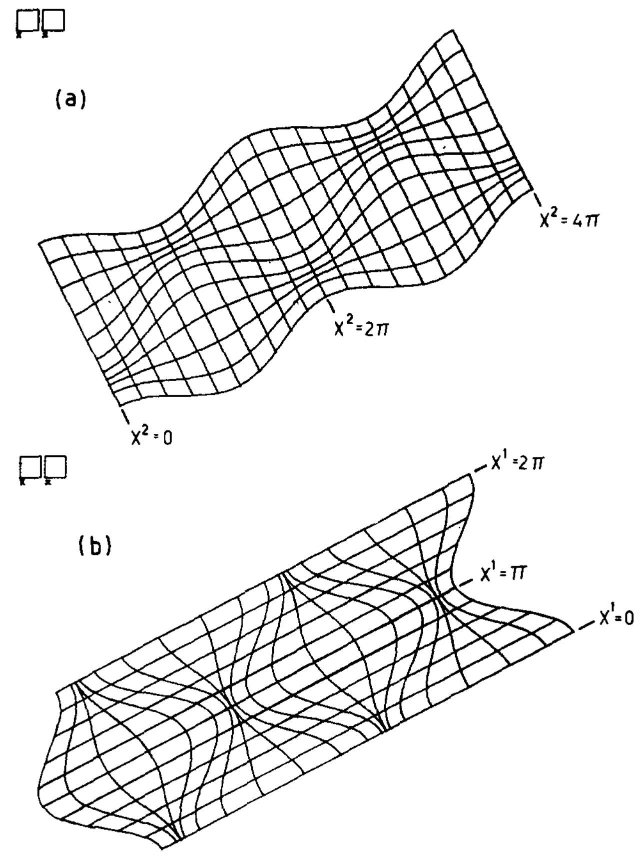

Fig. 16. Two solutions to strain related restrictions, based on similar starting values of $a_{12} = 0.5$, $a_{21} = 0.75$, $a_{22} = 1$, $A_{111} = 0.5$, A_{211} as the unknown complementary harmonic. (a) was solved for neutrality along $X^2 = n\pi$, resulting in $A_{211} = -0.25$ and $a_{21} = -0.375$, whilst (b) was solved for isochorism on $X^1 = n\pi$, giving $A_{211} = 1$ and $a_{21} = -0.5$. The range of both plots is $(0 \leqslant X^1 \leqslant 2\pi, 0 \leqslant X^2 \leqslant 4\pi)$.

layers of X^2 = constant remain straight, over the entire transformation! Starting with the same values and solving for isochorism along $X^1 = 0$ yields $a_{21} = -0.5$ and $A_{211} = 1$, with the same remarkable geometrical consequence (Fig. 16b).

Other restrictions of shear strain neutrality and of isochorism have been considered with harmonic combinations of the form A with B, C with D, etc. The restrictions invariably involve more and simpler equations, reflecting the lower harmony between the geometrical identities introduced by harmonics of the form A or D than those of the form B or C. In general the restrictions become very simple on deletion of the simple shear terms. The case of isochorism applied on a combination of A_{111} and C_{211}, can be solved by any values of the harmonics for isochorism along $X^1 = 0$ when simple shear terms are deleted. Figure 17 shows such a transformation, given by $a_{11} = 1.333$, $a_{22} = 0.75$, $A_{111} = 0.75$ and $C_{211} = 0.25$.

As with previous combinations, general isochorism and shear strain neutrality are not possible.

Combinations of mixed harmonics on one axis only

Combination of two mixed harmonics, defining pinch and swell on two axes, will not, according to the preceding section, give cases of general isochorism or general angular shear strain neutrality. In this section some cases of general isochorism will be shown to exist for combinations of two pinch and swell variations on the same axis.

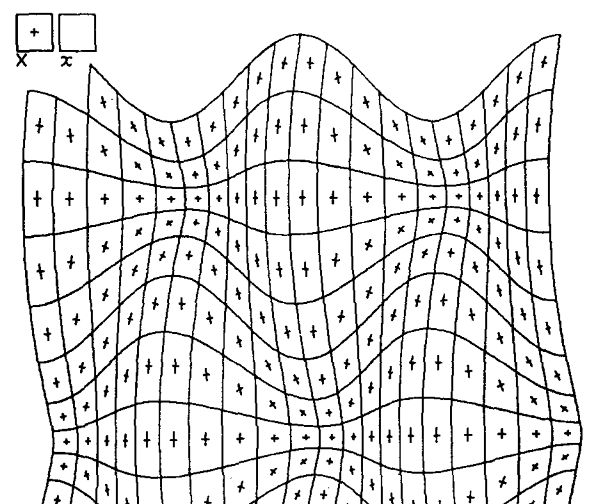

Fig. 17. A transformation in which isochorism is achieved along $X^1 = n\pi$ by using the harmonics A_{111} and C_{211}. An overall pure shear is present, the transformation being given by: $A_{111} = 0.75$, $C_{211} = 0.25$, $a_{11} = 1.333$ and $a_{22} = 0.75$. It is plotted over the range $(0 \leqslant X^1 \leqslant 2\pi, 0 \leqslant X^2 \leqslant 4\pi)$.

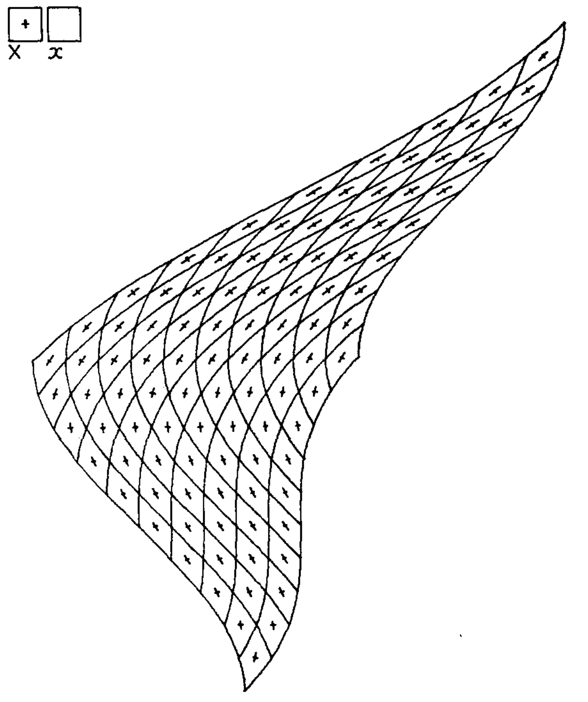

Fig. 18. A portion of a generally isochoric transformation generated using two mixed harmonics on the same axis and a homogeneous simple shear. The range of the plot is $(0 \leq X^1 \leq \pi, 0 \leq X^2 \leq \pi)$, the origin being the point plotted lowest on the page. The transformation is: $A_{211} = 1, D_{211} = -1, a_{11} = a_{12} = a_{22} = 1$ and $a_{21} = 0$.

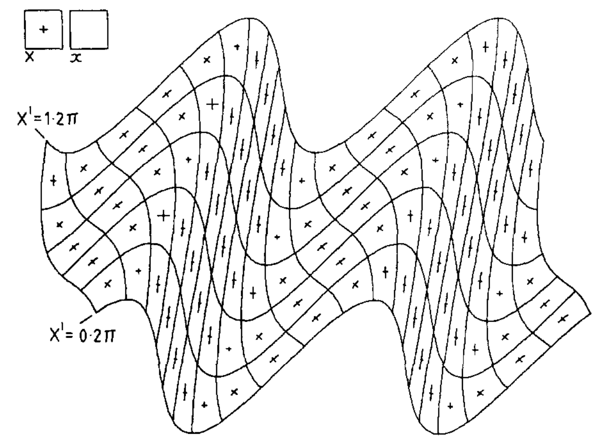

Fig. 19. A generally isochoric transformation using two mixed harmonics on each axis. The parameters are: $A_{111} = D_{111} = 1.125$, $A_{211} = D_{211} = 0.5$, $a_{11} = 1.5$ and $a_{22} = 0.667$. Plotted in the range $(0.2\pi \leq X^1 \leq 1.2\pi, 0 \leq X^2 \leq 4\pi)$.

Restrictions of shear strain neutrality and of isochorism have been investigated with combinations of the form of A_{211} with D_{211}, etc. All variations are given for the x^2 axis only. Obviously, a similar set of results would apply for variation on x^1.

General shear strain neutrality is excluded by all combinations in that a sum-of-squares term appears in each set of restrictive equations, as do equations with single terms only.

General isochorism is possible in the case depicted in Fig. 18 which is a combination of A_{211} with D_{211}. A similar result would be expected for B_{211} with C_{211}, as well as for equivalent combinations on x^1.

In Fig. 18 a solution is plotted, in which unitary values of a_{11}, a_{12} and a_{22} have been used, together with opposite values of A_{211} and D_{211} (1, and -1, respectively). Only half a wavelength has been plotted in either dimension.

Combination of all mixed harmonics

To complete the examination of mixed harmonic transformations, a general combination of all possible first-order mixed harmonics is now presented.

Restrictions of shear strain neutrality and of isochorism have been examined with combinations of all harmonics with indices $i11$, without simple shear terms a_{12} and a_{21}. These restrictions incorporate the individual groups of restrictions on non-simple-shear mixed harmonic transformations presented in preceding sections, to which they can be reduced by deletion of appropriate harmonics.

General neutrality is not, of course, to be expected. Shear strain neutral surfaces will be common, and represent combinations of the form described previously. No further examples are presented here.

Computer solutions have indicated that all harmonics A and D are interrelated, as are all in the B/C group, by these restrictions. Generally isochoric transformations can be made up by using all of the A and D harmonics, or all the B and C, or all harmonics together. The strong interrelationship explains the absence of general isochorism in the combinations examined in previous sections. It means that, for a specification of one of the pure shear components and one of the harmonics from each group, all terms become specified if isochorism is to be general.

For given a_{11} and a_{22} and a single given harmonic from one of the interrelated groups, all but one of the terms in the transformation using that group are uniquely defined. This term is defined as the positive or negative value of a square root.

A solution using the A/D group is illustrated in Fig. 19, the transformation being defined by $A_{111} = D_{111} = 1.125$, $A_{211} = D_{211} = 0.5$, $a_{11} = 1.5$ and $a_{22} = 0.6667$. Despite the fact that only 'pinch-and-swell' harmonics are used, the lines of X^1 = constant could still represent layering traces in similar folds. These harmonic restrictions thus force the mixed harmonics into combinations in which they simulate 'fold' development with extreme layer-parallel angular shear.

Two solutions using all harmonics are illustrated in Fig. 20. Figure 20(a) uses unitary pure shear components, and has harmonic values of $A_{111} = B_{111} = C_{111} = D_{211} = 1, A_{211} = B_{211} = C_{211} = D_{111} = -1$. In Fig. 20(b) there is an overall pure shear given by $a_{11} = 1.25$ and $a_{22} = 0.8$, and harmonic terms of lower amplitude are used.

The general pattern in all of these examples is similar, isochorism at each point being maintained by the balancing of the pinch and swell of co-existing harmonics to give 'similar fold' surface relationships with strong 'interlayer shear'. Increasing the ratios of individual harmonics within groups increases the asymmetry of the structures, as is evident in Fig. 20.

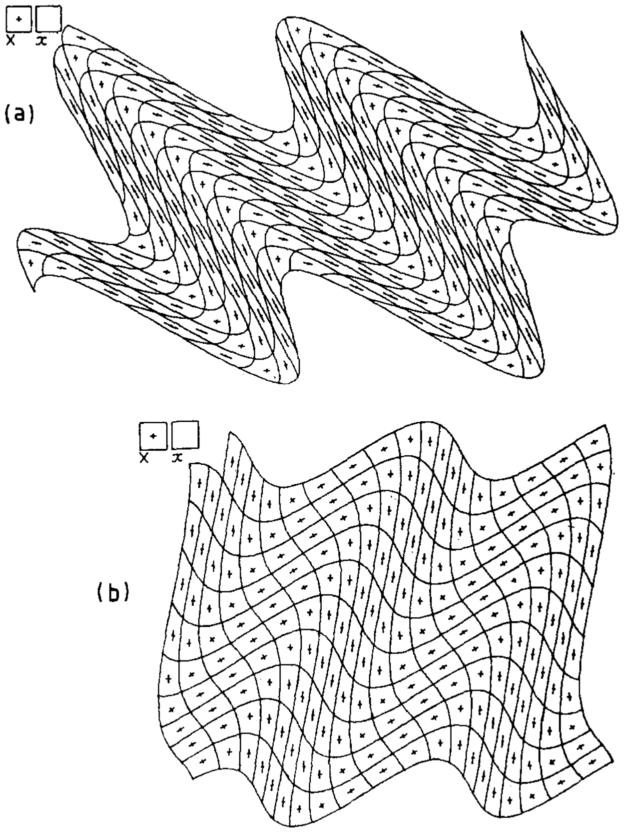

Fig. 20. Generally isochoric transformations involving all first-order mixed harmonics: (a) $A_{111} = B_{111} = C_{111} = D_{211} = 1$, $A_{211} = B_{211} = C_{211} = D_{111} = -1$; $a_{11} = a_{22} = 1$; (b) $A_{111} = B_{111} = D_{111} = 0.5$; $A_{211} = B_{211} = D_{211} = 0.32$; $C_{111} = -0.5$; $C_{211} = -0.32$; $a_{11} = 1.25$ and $a_{22} = 0.8$. Both transformations are plotted in the range $(0 \leqslant X^1 \leqslant 2\pi, 0 \leqslant X^2 \leqslant 4\pi)$.

Cosine curve with all mixed harmonics

It is paradoxical that the quest for mixed harmonics to produce pinch and swell with general isochorism and local shear strain neutrality should result in fold-like patterns. It makes reintroduction of the cosine curve appear of questionable value.

Writing out the relationships for isochorism and shear strain neutrality indicates that this is indeed so. The mixed harmonic restrictions are still present, with the A_{101} appearing in new single term equations only. This makes the system insoluble in a general sense. It is not possible to achieve general isochorism in a transformation in which a basic form given by a first-order geometric function curve is to receive additional layer-parallel heterogeneous strain through first-order mixed harmonics only.

To produce variations as shown in Fig. 11, which was derived by experiment using interactive graphics, other harmonics, including ones describing second-order axially inhomogeneous pure shear, are used. The restrictions to a number of such combinations have been examined, but no generally isochoric solutions have emerged.

SUMMARY

A plane strain coordinate transformation based on a multiple harmonic series has been presented, and shown to be capable of modelling a wide range of deformation patterns although the patterns with isochorism at each point appear to be restricted. Its effects can be split up into homogeneous components, termed a_{iK}, and harmonic terms A_{imn}, B_{imn}, C_{imn} and D_{imn} applied simultaneously.

The deformation expressed in the coordinate transformations is capable of representing all styles of folding independently of the mechanism of folding and can represent a wide range of other inhomogeneous deformations as well.

Of the various terms in the deformation,

- a_{iK} terms represent the components of a homogeneous deformation, so that by varying these quantities, various amounts of homogeneous shortening, extension or shearing may be introduced.
- A_{i00} terms represent a homogeneous translation of the deformed body.
- B_{im0}, C_{i0n}, D_{i0n}, D_{im0} terms are redundant since they are associated with terms involving sin (0).
- A_{10n} and B_{10n} terms, when non-zero, represent inhomogeneous shear along X^1 whereas
- B_{2m0} and C_{2m0} terms, when non-zero, represent inhomogeneous shear along X^2.

Other terms can lead to an inhomogeneous pure shear. Thus with A_{20n} and B_{20n} non zero and B_{1m0}, C_{1m0} zero, the relative distance between X^1 = constant planes is altered in an inhomogeneous manner.

- A_{imn} terms with i, m and n non-zero simulate a 'pinch-and-swell' type of deformation.

An alternative way of viewing the terms in the transformation is to consider their significance in simulating a fold shape where X^1 is parallel to the axial plane trace and X^2 is parallel to the trace of initial layering:

- A_{10n} and B_{10n} terms represent the gross fold outline.
- A_{20n} and B_{20n} terms represent inhomogeneous pure shear along X^2.
- A_{1m0} and C_{1m0} terms represent axial plane extension variable from layer to layer but not along layers.
- A_{1mn}, B_{1mn}, C_{1mn} and D_{1mn} represent inhomogeneous across-layer extension including pinch and swell.
- A_{2mn}, B_{2mn}, C_{2mn}, D_{2mn} represent generally variable extension along layer traces, i.e. they contain most of the 'buckling' inhomogeneity.
- a_{iK} terms represent a homogeneous strain applied to the whole fold.

In this paper a number of constraints are developed for the transformation including a condition for constant volume (isochoric) deformation, and zero shear strain for lines initially normal to the distorted layer.

The nature of the multiple harmonic coordinate transformation precludes generally isochoric solutions from folding with inhomogeneous layer-parallel shear strain. A solution to a perfectly isochoric buckling may have to resort to solutions based on multiple angle formulae going to high orders of harmonics.

Simple numerical relationships governing the selection of harmonics can be found such that isochorism and shear strain neutrality occur along selected surfaces of a transformation. A number of examples of these have been presented as results of semi-automatically executed derivations. The basis of these derivations has been a logical rather than a numerical program.

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